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HOMOTOPY LOCALIZATION FUNCTOR L_f WITH RESPECT TO MAPS f HAVING A WEDGE OF SPHERES AS HOMOTOPY COFIBRE

BY

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ABSTRACT

In the connected case, we compute explicitly the f-localization (in the sense of [3]) for the class of maps $Z^{(n)} \hookrightarrow Z$ in which the cofibre is a wedge of spheres. We have an analogous result over the rationals where the cofibre is arbitrary.

1. Introduction

The homotopy localization framework described by Farjoun in [3] gives a general way to do homotopy theory which encompasses all the other major localization theories, e.g., Sullivan–Quillen localizations, Bousfield–Kan localizations (when R_∞ is idempotent), Bousfield homological localization, and Quillen's plus construction.

For any map $f: A \to B$, Farjoun constructs an associated localization theory around an idempotent functor L_f and shows that the above theories correspond to specific choices of f.

Thus, given a map f, it becomes imperative to be able to compute L_fX. Unfortunately this task is not an easy one since, in general, the construction involves a transfinite induction (Quillen's small object argument is involved).

In this paper, in the category of connected CW-complexes, we compute L_fX for an important class of maps f, i.e., inclusions, $Z^{(n)} \hookrightarrow Z$, of nilpotent nth skeletons $Z^{(n)}$ into nilpotent spaces Z for which the cofibre is a wedge of spheres

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(e.g. $S^2 \vee S^2 \hookrightarrow S^2 \times S^2$). In fact we show (Theorem 9) that $L_f X$ depends only on $[Z^{(n)}, X]$ and $Z^{(k)}$, where $k-1$ is the connectivity of the homotopy cofibre, C_f, of the map f ($k > n$). More precisely, $L_f X$ is the $(k-1)$-Postnikov stage of the adjunction space

$$X \cup (\bigcup_{\phi\circ h} (\bigcup_{\alpha} e^k))_{\phi \in I}$$

where $h\colon \vee_\alpha S^{k-1} \to Z^{(n)}$ is the attaching map of the kth cells into $Z^{(n)}$, and $I \subset map(Z^{(n)}, X)$ is a set containing exactly one representative of each homotopy class in $[Z^{(n)}, X]$ such that $\phi \circ h \not\simeq *$.

In the case of $\mathcal{P}$-localisation, in the sense of [4], we show that a $\mathcal{P}$-local space X is f-local if and only if it is $f_{\mathcal{P}}$-local (Theorem 12). Thus f-localization can be done inside the $\mathcal{P}$-category.

Finally, in the category of simply connected rational spaces, we obtain the same result as the integral one except that the condition on the cofibre is lifted, i.e., the inclusions $Z^{(n)} \hookrightarrow Z$ are arbitrary (Theorem 16).

2. Definitions

We work in the category of topological spaces (*Top*) having the homotopy type of a CW-complex. The pointed category will be denoted by Top_*, and the respective function complexes $map(X, Y)$ and $map_*(X, Y)$ will be denoted by Y^X and $(Y^X)_*$.

Definition 1: A space X is f-local, where $f\colon A \to B$ is a map between cofibrant spaces, if the induced map on function complexes

$$f_\#\colon X^B \longrightarrow X^A$$
$$g \mapsto g \circ f$$

is a weak equivalence.

Remark: If we assume that all spaces are cofibrant, one could define a pointed version of this notion in the obvious way. In the connected category (Top_o) the commutative diagram

$$\begin{array}{ccccc} (X^B)_* & \longrightarrow & X^B & \xrightarrow{ev} & X \\ \big\downarrow{\scriptstyle (f_\#)_*} & & \big\downarrow{\scriptstyle f_\#} & & \big\downarrow{\scriptstyle =} \\ (X^A)_* & \longrightarrow & X^A & \xrightarrow{ev} & X \end{array}$$

of fibrations and its induced long exact sequence in homotopy together with the 5-lemma show that $(f_\#)_*$ is a weak equivalence if and only if $f_\#$ is.

Definition 2: A functor $\mathcal{F}$ is said to be coaugmented if it comes equipped with a natural transformation $j\colon Id \to \mathcal{F}$. The functor is said to be idempotent if both natural maps $j_{\mathcal{F}X}$ and $\mathcal{F}(j_X)$ are weak equivalences and are homotopic to each other. We say that the coaugmentation map j_X is universal with respect to maps $X \to T$ into f-local spaces T if such a map factors uniquely up to homotopy through j_X.

THEOREM 1 ([3]): *For any map $f\colon A \to B$ in Top (or Top_*) there exists a functor L_f from Top (or Top_*) to f-local spaces, called the f-localization functor, which is coaugmented and idempotent. Any two such functors are naturally weakly equivalent to each other. Moreover, the coaugmentation is universal.*

Remark: The functors $L_{W\to *} = L_{*\to W}$ will be denoted by P_W. The functor P_W is called the W-nullification functor.

COROLLARY 2 ([3]): *$L_f \simeq L_g$ if and only if every f-local space is also g-local and vice versa.*

3. Characterizations in the connected case

For now on, we will work exclusively in Top_*, thus we will forego the subscript for the base point preserving function complex $(X^Y)_*$. Consider well pointed spaces (X, x_o), (Y, y_o), and (Z, z_o). Recall that the space $X \ltimes Y$ is the quotient space $(X \times Y)/(X \times \{y_o\})$. Let $\phi \in Z^Y$ and consider the pointed space (Z^Y, ϕ). The pointed exponential law is thus given by

$$map_*(X, Z^Y) \quad \cong \quad \{h\colon X \ltimes Y \to Z | h_{|Y} = \phi\}.$$

This in turn induces on the level of pointed homotopy classes the following equivalence,

$$[(X, x_o), (Z^Y, \phi)] \quad \cong \quad [X \ltimes Y, Z]_\phi,$$

where the subscript ϕ is there as a reminder that homotopies are taken relative to ϕ.

PROPOSITION 3: *A connected space (X, x_o) is f-local if and only if for all $n \geq 1$ and for all $\phi \in X^B$*

- $[B, X] \cong [A, X]$, *and*
- $[S^n \ltimes B, X]_\phi \cong [S^n \ltimes A, X]_{\phi\circ f}$.

Proof: This follows from the above observations. ∎

Remark: A connected W-null space X is a space which satisfies in particular

$$[W, X] \cong *,$$

i.e., the space X^W is connected. Thus, without lost of generality, the base point ϕ of X^W can be chosen to be the constant map at x_o. This implies that the higher dimensional conditions all collapse to

$$0 \cong [S^n \ltimes W, X]_{x_o} \cong [S^n \wedge W, X] \cong [\Sigma^n W, X].$$

Example: A connected S^n-null space X is just an arbitrary Postnikov $(n-1)$-stage since it has no homotopy groups above dimension $(n-1)$. We have, in the connected case, $P_{S^{n+1}} \simeq P_n$, the nth Postnikov section functor.

In general, half-smashes are not very tractable. But in the special case where $f: A \to B$ is the cofibre of a map $g: Z \to A$ we have

PROPOSITION 4: *If $f: A \to B$ is the cofibre of a map $g: Z \to A$, a connected space X is f-local if and only if*

- *for all $n \geq 1$, $[\Sigma^n Z, X] = 0$, and*
- *the map $[f, X]: [B, X] \to [A, X]$ is onto.*

Proof: Given any map $\phi \in X^B$ we have a fibration

$$(X^B, \phi) \to (X^A, \phi \circ f) \to (X^Z, *),$$

and a long exact homotopy sequence

$$[\Sigma^{n+1} Z, X] \to \pi_n(X^B, \phi) \to \pi_n(X^A, \phi \circ f) \to [\Sigma^n Z, X].$$

Therefore, the middle arrow is an isomorphism for all $n \geq 0$ if and only if $[\Sigma^n Z, X] = 0$, $n \geq 1$, and $[f, X]$: $[B, X] \to [A, X]$ is onto. Notice that we use the natural coaction $B \to \Sigma Z \vee B$ to show that the map $[f, X]$ is injective ([5]). ∎

For an arbitrary map $f: A \to B$ we deduce the following necessary conditions on a connected space X to be f-local.

PROPOSITION 5: *If X is f-local then*

- *for all $n \geq 0$, $[\Sigma^n C_f, X] = 0$, where C_f is the homotopy cofibre of the map $f: A \to B$, and*

- *the map* $[f, X]\colon [B, X] \to [A, X]$ *is onto.*

Proof: By considering the connected component of the constant map in X^B, the result follows from Proposition 3 and the long exact sequence of pointed sets ([6, p. 134])

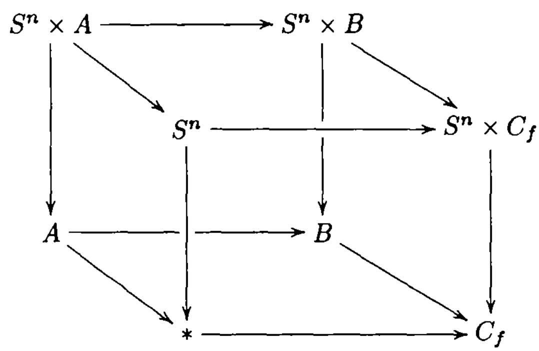
$$[\Sigma^n C_f, X] \to [\Sigma^n B, X] \to [\Sigma^n A, X] \to [\Sigma^{n-1} C_f, X]. \qquad \blacksquare$$

Remark: Any f-local space is thus C_f-null. The reverse implication is not true as we will see in the next section.

To detect f-local spaces, Casacuberta and Rodriguez showed recently in [1] that one can forget the map ϕ in Proposition 3 and consider only base point preserving homotopy classes, namely

PROPOSITION 6 ([1]): *Let* $f\colon A \to B$ *be any map between CW-complexes. Then a space* X *is* f*-local if and only if* f *induces a bijection* $[B, X] \cong [A, X]$ *together with* $[S^n \ltimes B, X] \cong [S^n \ltimes A, X]$ *for* $n \geq 1$.

Consider the following commutative diagram



where the bottom face is a homotopy pushout and all vertical faces are homotopy pullbacks induced by the trivial fibration $S^n \times C_f \to C_f$. Then, by a result of Cornea ([2]), the induced diagram

$$\begin{array}{ccc} S^n \ltimes A & \longrightarrow & S^n \ltimes B \\ \downarrow & & \downarrow \\ * & \longrightarrow & S^n \ltimes C_f \end{array}$$

is a homotopy pushout. Thus the homotopy cofibre of the map $S^n \ltimes f$ is $S^n \ltimes C_f$. We get a sequence for each $n \geq 1$,

$$[S^n \ltimes C_f, X] \to [S^n \ltimes B, X] \to [S^n \ltimes A, X],$$

which is exact (as pointed sets) at the middle term. If X is C_f-null, then X^{C_f} is connected. Thus for each $\psi \in X^{C_f}$ and each $n \geq 1$,

$$[S^n \ltimes C_f, X]_\psi \cong [\Sigma^n C_f, X] \cong 0,$$

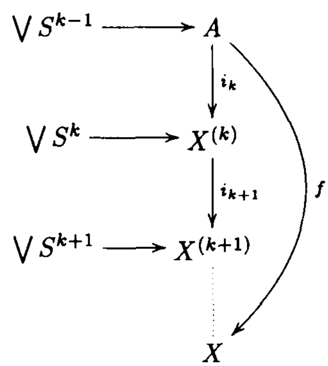
which implies that $[S^n \ltimes C_f, X] \cong *$ for each $n \geq 1$. In the case where A and B are co-H-spaces, the above remarks together with Proposition 6 show that

PROPOSITION 7: *A connected space X is f-local if and only if*

- *X is C_f-null,*
- *the map $[f, X]\colon [B, X] \to [A, X]$ is onto, and*
- *each map $[S^n \ltimes f, X]\colon [S^n \ltimes B, X] \to [S^n \ltimes A, X]$ is onto for $n \geq 1$.*

4. Main theorem

Let (X, A) be a relative CW-complex in which both X and A are nilpotent, and consider the inclusion $f\colon A \hookrightarrow X$. Let $k = \inf\{l | X^{(l)} \neq A\}$. Without loss of generality we can assume that $k = conn(C_f) + 1$. We thus have a sequence of cofibration given by the attaching maps



such that $f = \cdots \circ i_{k+1} \circ i_k$. In general, if $f = g \circ h$ and Y is g- and h-local, then it is f-local. With the above notation we have

THEOREM 8: *If C_f is a wedge of spheres, then Y is f-local if and only if it is i_k-local.*

Proof: If Y is i_k-local then, since i_k is the cofibre of the attaching map $\vee S^{k-1} \to A$, by Proposition 4 we have $\pi_{\geq k}(Y) = 0$. Thus for each $l \geq k$ the maps $[i_l, Y]$ are onto. It follows that Y is i_l-local for each $l \geq k$. Hence it is f-local. If Y is f-local, then for each map $\phi \in Y^{X^{(k)}}$, the induced maps

$$\pi_n(Y^{X^{(k)}}, \phi) \to \pi_n(Y^A, \phi \circ i_k)$$

are onto for all $n \geq 0$. Thus, the long exact sequence in homotopy associated to the fibration sequence $Y^{X^{(k)}} \to Y^A \to Y^{\vee S^{k-1}}$ gives rise to short exact sequences

$$0 \to [\bigvee S^{n+k}, Y] \to \pi_n(Y^{X^{(k)}}, \phi) \to \pi_n(Y^A, \phi \circ i_k) \to 0$$

for each $n \geq 0$. But Y, being f-local, is C_f-local. By hypothesis C_f is a wedge of spheres of connectivity $k-1$. Thus $\pi_{\geq k}(Y) = 0$, and hence Y is i_k-local. ∎

THEOREM 9: *Let Z be a nilpotent connected CW-complex such that the inclusion $f\colon Z^{(n)} \hookrightarrow Z$ has as cofibre a non-trivial wedge of spheres together with $Z^{(n)}$ nilpotent. Let $k = conn(C_f) + 1$. Then for any connected space X*

$$L_f X \simeq P_{S^k}\{X \cup (\bigcup_{\phi \circ h} (\bigcup_{\alpha} e^k))_{\phi \in I}\},$$

where $h\colon \bigvee_\alpha S^{k-1} \to Z^{(n)}$ is the attaching map of the kth cells into $Z^{(n)}$, and $I \subset map(Z^{(n)}, X)$ is a set containing exactly one representative ϕ of each homotopy class in $[Z^{(n)}, X]$ such that $\phi \circ h \not\simeq *$.

Proof: By Theorem 8 and Corollary 2, $L_f \simeq L_i$, where i is the inclusion $Z^{(n)} \hookrightarrow Z^{(k)}$. Note that $n < k$. From Proposition 4, any connected i-local space W has $\pi_{k+j} W = 0$ for all $j \geq 0$. Moreover, any map $Z^{(n)} \to W$ extends through i to a map $Z^{(k)} \to W$. Consider the following homotopy pushout diagram

$$\begin{array}{ccccc} \bigvee_I (\bigvee_\alpha S^{k-1}) & \xrightarrow{\vee h} & \bigvee_{\phi \in I} Z^{(n)} & \xrightarrow{\vee_{\phi \in I} \phi} & X \\ & & \downarrow & & \downarrow \\ & & \bigvee_I Z^{(k)} & \longrightarrow & \widehat{X} \end{array}$$

where $\widehat{X} = X \cup (\bigcup_{\phi \circ h} (\bigcup_\alpha e^k))_{\phi \in I}$. The space $P_{S^k} \widehat{X}$ is i-local since any map $Z^{(n)} \to P_{S^k} \widehat{X}$ extends through i to a map $Z^{(k)} \to P_{S^k} \widehat{X}$. Indeed this is clear because by construction $(P_{S^k} \widehat{X})^{(n)} = X^{(n)}$. Thus, from the cellular approximation theorem, any map $Z^{(n)} \to P_{S^k} \widehat{X}$ is homotopic to a map $Z^{(n)} \to X^{(n)} \subset X$. Clearly there is no problem to extend the map to $Z^{(k)}$ by definition of $\widehat{X}$. If we use the functor $map(-, W)$ on the defining diagram of $\widehat{X}$, we obtain a homotopy pullback diagram

$$\begin{array}{ccc} W^{\widehat{X}} & \longrightarrow & W^X \\ \downarrow & & \downarrow \\ W^{\bigvee Z^{(k)}} & \xrightarrow{\simeq} & W^{\bigvee Z^{(n)}} \end{array}$$

in which the bottom arrow is a weak equivalence since W is i-local. Thus the top arrow is also a weak equivalence, and consequently it induces an isomorphism on the set of path components, i.e., $[\widehat{X}, W] \cong [X, W]$. Moreover, W being S^k-null, we have by universality

$$[P_{S^k}\widehat{X}, W] \cong [\widehat{X}, W] \cong [X, W],$$

and the result follows, namely $L_f X \simeq P_{S^k}\widehat{X}$. ∎

Remarks:

- For a general $f\colon A \to B$ with its cofibre C_f still a wedge of spheres, any f-local space X has, by Proposition 5, $\pi_{>l}X = 0$, where l is the connectivity of the cofibre C_f. Moreover, if $l \leq conn(A)$, then $L_f \simeq P_{S^{l+1}}$.
- If $[Z^{(n)}, X]$ has a group structure induced by either $Z^{(n)}$ being a co-H-space or X being an H-space, then one can restrict I, in Theorem 9, to generators.

Example: Consider the following inclusion $f\colon S^2 \vee S^2 \hookrightarrow S^2 \times S^2$. Its homotopy cofibre is S^4. Moreover, f is the cofibre of the map $[i_1, i_2]\colon S^3 \to S^2 \vee S^2$, where the i_j are the inclusions of S^2 into respectively the first and second factor of $S^2 \vee S^2$, and $[,]$ denotes the Whitehead product ([6, p. 472]). Thus, from Proposition 4, a connected space X is f-local if and only if for all $n \geq 1$, $\pi_{3+n}(X) = 0$, and each map $\phi\colon S^2 \vee S^2 \to X$ extends through f to a map $S^2 \times S^2 \to X$. This last condition amounts to an obstruction problem. Consider the following diagram

$$\begin{array}{ccccc} S^3 & \xrightarrow{[i_1,i_2]} & S^2 \vee S^2 & \xrightarrow{\phi} & X \\ \big\downarrow & & \big\downarrow f & \nearrow & \\ D^4 & \longrightarrow & S^2 \times S^2 & & \end{array}$$

From elementary obstruction theory there is a dotted arrow completing this diagram for each ϕ if and only if the maps $\phi \circ [i_1, i_2] \simeq *$, i.e., if and only if $[\alpha, \beta] = 0$ for all $\alpha, \beta \in \pi_2(X)$. We can now calculate $L_f S^2$. Given a map $\phi \in map(S^2 \vee S^2, S^2)$, the corresponding class $\overline{\phi \circ [i_1, i_2]} \in \pi_3 S^2$ is a multiple of the class $[\iota, \iota]$ where $\iota = \overline{Id_{S^2}}$. Consider the pushout diagram

$$\begin{array}{ccccc} S^3 & \xrightarrow{[i_1,i_2]} & S^2 \vee S^2 & \xrightarrow{Id \vee Id} & S^2 \\ & & \big\downarrow & & \big\downarrow \\ & & S^2 \times S^2 & \longrightarrow & \underbrace{S^2 \cup_{[\iota,\iota]} e^4}_{\widehat{S^2}} \end{array}$$

We claim that the third Postnikov section of this space, i.e., $P_{S^4}\widehat{S^2}$, is f-local. But this is clear since $\pi_{\geq 4}(P_{S^4}\widehat{S^2}) = 0$, and $(P_{S^4}\widehat{S^2})^{(2)} \simeq S^2$. Thus, by the cellular approximation theorem, any map $\psi\colon S^2 \vee S^2 \to P_{S^4}\widehat{S^2}$ is homotopic to a map $\phi\colon S^2 \vee S^2 \to S^2$. Theorem 9 shows that it is the right candidate, i.e.,

$$L_f S^2 \simeq P_{S^4}(S^2 \cup_{[\iota,\iota]} e^4).$$

Moreover, we can calculate the homotopy groups of $L_f S^2$. Notice that the homotopy class $[\iota, \iota] \in \pi_3 S^2$ is twice the one of the Hopf map. Thus

$$\pi_k(L_f S^2) \cong \begin{cases} 0, & k = 1 \text{ or } k \geq 4, \\ \mathbb{Z}, & k = 2, \\ \mathbb{Z}/2\mathbb{Z}, & k = 3. \end{cases}$$

Finally, an easy calculation gives $L_f S^k$ for all $k \geq 1$, i.e.,

$$L_f S^k \simeq \begin{cases} S^1, & k = 1, \\ P_{S^4}(S^2 \cup_{[i_1,i_2]} e^4), & k = 2, \\ P_{S^4} S^3 \simeq K(\mathbb{Z}, 3), & k = 3, \\ *, & k \geq 4. \end{cases}$$

5. The $\mathcal{P}$-local case

Recall that if $\mathcal{P}$ is a subset of Π, the set of prime numbers, then a group G is said to be $\mathcal{P}$-local if the map $x \mapsto x^p$ is a bijection $\forall x \in G$ and $\forall p \in \Pi - \mathcal{P}$. For a simply connected space X we have

THEOREM 10 ([4]): *The following are equivalent.*

- *For all $n \geq 2$, $\pi_n X$ is $\mathcal{P}$-local.*
- *For all $n \geq 2$, $H_n(X)$ is $\mathcal{P}$-local.*

If a simply connected space X satisfies one of the above conditions, then it is said to be $\mathcal{P}$-local. For example, a rational space is a $\emptyset$-local space, or equivalently, all its homotopy groups and homology groups are $\mathbb{Q}$-vector spaces. Consider the map

$$f^{\mathcal{P}} : \bigvee_{p \in \Pi - \mathcal{P}} S^2 \xrightarrow{\vee p} \bigvee_{p \in \Pi - \mathcal{P}} S^2$$

where each map $p\colon S^2 \to S^2$ is a map of degree p. Then a simply connected space X is $\mathcal{P}$-local if and only if it is $f^{\mathcal{P}}$-local. In general $L_f(\Sigma X) \not\simeq \Sigma(L_f X)$ as the following example shows. Consider $X = K(\pi, n)$. On the one hand $P_{S^{n+1}} X = X$ and $\Sigma X \not\simeq *$, but on the other $P_{S^{n+1}}(\Sigma X) \simeq *$.

PROPOSITION 11: *For any simply connected space X we have*

$$L_{f^{\mathcal{P}}}(\Sigma X) \quad \simeq \quad \Sigma L_{f^{\mathcal{P}}} X.$$

Proof: One just has to notice that $\Sigma L_{f^{\mathcal{P}}} X$ is already $\mathcal{P}$-local. Thus, by universality of the functor $L_{f^{\mathcal{P}}}$, we have a commutative diagram (up to homotopy)

$$\begin{array}{ccc} \Sigma X & \longrightarrow & \Sigma(L_{f^{\mathcal{P}}} X) \\ & \searrow \quad \nearrow{\scriptstyle \phi} & \\ & L_{f^{\mathcal{P}}}(\Sigma X) & \end{array}$$

in which ϕ is clearly a weak equivalence (observe that it is an isomorphism in homology with coefficients in $\mathbb{Z}_{\mathcal{P}}$). Finally, since we are working in the category of spaces having the homotopy type of a CW-complex, it is a homotopy equivalence. ∎

Remark: We will denote $L_{f^{\mathcal{P}}} X$ by $X_{\mathcal{P}}$ and the rationalization by X_o.

Notice that by an homology long exact sequence argument we have

$$H_*(S^n \ltimes X) \cong H_*(X \vee \Sigma^n X) \cong H_*(X) \oplus H_*(\Sigma^n X).$$

Thus if X is $\mathcal{P}$-local then so is $S^n \ltimes X$. Moreover, by an argument similar to Proposition 11, the localization map

$$S^n \ltimes X \to (S^n \ltimes X)_{\mathcal{P}}$$

is given by $S^n \ltimes j_X$, where j_X is the $\mathcal{P}$-coaugmentation.

THEOREM 12: *Given a map $g\colon A \to B$ between simply connected spaces and a $\mathcal{P}$-local space X, one has that X is g-local if and only if X is $g_{\mathcal{P}}$-local.*

Proof: By naturality of the functor $L_{f^{\mathcal{P}}}$, we have a commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{g} & B \\ {\scriptstyle j_A}\downarrow & & \downarrow{\scriptstyle j_B} \\ A_{\mathcal{P}} & \xrightarrow{g_{\mathcal{P}}} & B_{\mathcal{P}} \end{array}$$

where j_A and j_B are the $\mathcal{P}$-coaugmentations. This, in turn, induces a commutative diagram

$$\begin{array}{ccc} [B, X] & \xrightarrow{[g,X]} & [A, X] \\ {\scriptstyle [j_B,X]}\uparrow & & \uparrow{\scriptstyle [j_A,X]} \\ [B_{\mathcal{P}}, X] & \xrightarrow{[g_{\mathcal{P}},X]} & [A_{\mathcal{P}}, X] \end{array}$$

in which the vertical arrows are isomorphisms because of universality of the functor $L_{f^{\mathcal{P}}}$. Thus the top arrow is an isomorphism if and only if the bottom one is. Again, by naturality, for any map $\phi \in X^{B_{\mathcal{P}}}$ we have $\phi \circ g_{\mathcal{P}} \circ j_A = \phi \circ j_B \circ g$. Consider the commutative diagram

$$\begin{array}{ccc} [S^n \ltimes B, X]_{\phi \circ j_B} & \xrightarrow{[S^n \ltimes g, X]} & [S^n \ltimes A, X]_{\phi \circ j_B \circ g} \\ {\scriptstyle [S^n \ltimes j_B, X]} \uparrow & & \uparrow {\scriptstyle [S^n \ltimes j_A, X]} \\ [S^n \ltimes B_{\mathcal{P}}, X]_{\phi} & \xrightarrow{[S^n \ltimes g_{\mathcal{P}}, X]} & [S^n \ltimes A_{\mathcal{P}}, X]_{\phi \circ g_{\mathcal{P}}} \end{array}$$

Now we claim that the vertical arrows are isomorphisms. In fact, showing that they are isomorphisms is equivalent to showing that the induced maps $X^{A_{\mathcal{P}}} \to X^A$ and $X^{B_{\mathcal{P}}} \to X^B$ are weak equivalences. But, by a result of Casacuberta and Rodriguez ([1]), it is sufficient to show that the induced maps $[j_A, X]\colon [A_{\mathcal{P}}, X] \to [A, X]$ and $[S^n \ltimes j_A, X]\colon [S^n \ltimes A_{\mathcal{P}}, X] \to [S^n \ltimes A, X]$ are isomorphisms (similarly for B). Because of universality and the fact that X is $\mathcal{P}$-local, they are indeed isomorphisms. Thus the top arrow is an isomorphism if and only if the bottom one is, and the result follows from Proposition 3. ∎

6. The rational case

From now on, we will work exclusively in the category of simply connected rational spaces. The next proposition was known to W. Chacholsky, D. Stanley and the author. A thorough treatment (paper in preparation by these three authors) on closed classes should complete the picture.

PROPOSITION 13: *For any space A of connectivity $n > 0$, we have $P_A \simeq P_{S^{n+1}}$.*

Proof: From Proposition 3 we have: a space X is S^{n+1}-null if and only if $\pi_{>n} X = 0$. A space X is A-null if and only if for all $k \geq 0$, $[\Sigma^k A, X] = 0$. It is a well known fact that for any space Z, $\Sigma Z \simeq_{\mathbb{Q}} \bigvee_\alpha S^{n_\alpha}$. Thus the last condition reads $[A, X] = 0$ and $\pi_{>n+1} X = 0$. Given that $\pi_{>n+1} X = 0$, X is, in particular, S^{n+2}-null. Thus, the result follows from universality, namely

$$\begin{aligned} [A, X] &\cong [P_{S^{n+2}} A, X] \\ &\cong [K(\pi_{n+1} A, n+1), X] \\ &\cong [P_{S^{n+2}}(\bigvee_{rk(\pi_{n+1} A)} S^{n+1}), X] \\ &\cong [\bigvee_{rk(\pi_{n+1} A)} S^{n+1}, X], \end{aligned}$$

and Corollary 2. ∎

COROLLARY 14: *Given a map $f: A \to B$, an f-local space X has $\pi_{>l}X = 0$, where $l = conn(C_f)$. Moreover, if $l \le conn(A)$, then $L_f \simeq P_{S^{l+1}}$.*

Proof: Using Proposition 4, the proof is essentially the same as in Proposition 13. ∎

Example: Consider the Hopf map $h: S^3 \to S^2$. Its cofibre is $\mathbb{C}P^2$. Thus, for simply connected rational spaces, $L_h \simeq P_{S^2} \simeq *$. On the other hand, the next Hopf map $g: S^7 \to S^4$ has $\mathbb{H}P^2$ as cofibre, thus, over the rationals, $L_g \simeq P_{S^4}$.

THEOREM 15: *Let $f: A \to B$ be a map together with its associated Quillen minimal model $(\mathbb{L}V, d_A) \to (\mathbb{L}(V \oplus W), D)$ ([5]). Let $k = conn(C_f) + 1$ and denote by i the inclusion $(\mathbb{L}V, d_A) \to (\mathbb{L}(V \oplus W_{k-1}), D)$. Then, a rational space X is f-local if and only if it is i-local.*

Proof: The same proof as for Theorem 8 goes through except at the very end where we need $\pi_{\geq k}(X) = 0$. This is the case, since by Proposition 13, $P_{C_f} \simeq P_{S^k}$, and X is C_f-null. ∎

Finally, we have the rational version of Theorem 9.

THEOREM 16: *Let $f: A \to B$ be a map together with its associated Quillen minimal model $(\mathbb{L}V, d_A) \to (\mathbb{L}(V \oplus W), D)$. Suppose that $V_{>n} = 0$ for some n and that $k - 1 = conn(C_f) > n$, i.e., $W_{\leq n} = 0$. Given a minimal model $(\mathbb{L}X, d_Z)$ of a space Z, a model for $L_f Z$ is given by the $(k-1)$th Postnikov section of the space corresponding to the adjunction model*

$$\left(\mathbb{L}\Big(X \bigoplus_{\phi \in I} (\bigoplus_{\alpha} < z_\alpha^\phi >)\Big), \delta\right)$$

where

- *for each ϕ, $\{z_\alpha^\phi\}$ is in one-to-one correspondence with a basis $\{w_\alpha\}$ of W_{k-1},*
- *$I \subset Hom(\mathbb{L}V, \mathbb{L}X)$ is a set containing exactly one representative ϕ of each homotopy class in $[\mathbb{L}V, \mathbb{L}X]$ such that $\overline{\phi D w_\alpha} \neq 0$ for some α, and*
- *the differential is given by $\delta|_X = d_Z$ and $\delta z_\alpha^\phi = \phi D w_\alpha$.*

Proof: Consider the inclusion $i: (\mathbb{L}V, d_A) \to (\mathbb{L}(V \oplus W_{k-1}), D)$. Then, by Theorem 15 and Corollary 2, $L_f \simeq L_i$. But then i is a map satisfying the hypotheses of Theorem 9 since its cofibre is a wedge of spheres. ∎

Remark: The last two results are in sharp contrast with the integral ones since we do not impose on C_f to be a wedge of spheres.

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References

[1] C. Casacuberta and J. L. Rodriguez, *On weak homotopy equivalences between mapping spaces*, Topology **37** (1996), 709–717.

[2] O. Cornea, *Cone-length and Lusternik-Schnirelmann category*, Topology **33** (1996), 95–111.

[3] E. D. Farjoun, *Cellular Spaces, Null Spaces and Homotopy Localization*, Lecture Notes in Mathematics **1622**, Springer-Verlag, Berlin, 1996.

[4] P. J. Hilton, G. Mislin and J. Roitberg, *Localization of Nilpotent Groups and Spaces*, North-Holland, Amsterdam, 1975.

[5] D. Quillen, *Rational homotopy theory*, Annals of Mathematics (2) **90** (1969), 205–295.

[6] R. M. Switzer, *Algebraic Topology—Homotopy and Homology*, Springer-Verlag, Berlin, 1975.

[7] G. W. Whitehead, *Elements of Homotopy Theory*, Graduate Texts in Mathematics **61**, Springer-Verlag, Berlin, 1978.